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# On 3-dimensional terminal singularities

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Introduction. Canonical and terminal singularities are introduced by M. Reid [4], [5]. He proved that 3-dimensional terminal singularities are cyclic quotient of smooth points or cDV points [5].

Let  $(X, p)$  be a 3-dimensional terminal singularity of index  $m$  with the associated  $\mathbf{Z}_m$ -cover  $(\tilde{X}, \tilde{p}) \rightarrow (X, p)$ . If  $(\tilde{X}, \tilde{p})$  is smooth, then it is known as Terminal Lemma (Danilov [2], D. Morrison-G. Stevens [3]) that there exist an integer  $a$  prime to  $m$  and coordinates  $x, y, z$  of  $(\tilde{X}, \tilde{p})$  which are  $\mathbf{Z}_m$ -semi-invariants such that  $\sigma(x) = \zeta x$ ,  $\sigma(y) = \zeta^{-1}y$ ,  $\sigma(z) = \zeta^a z$  for the standard generator  $\sigma$  of  $\mathbf{Z}_m$ , where  $\zeta$  is a primitive  $m$ -th root of 1. In this paper, we consider the case where  $(\tilde{X}, \tilde{p})$  is a singular point and  $m > 1$ . The main results are Theorems 12, 23, 25 and Remarks 12.2, 23.1, 25.1. These, together with the Terminal Lemma above, almost classifies 3-dimensional terminal singularities.

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## §1. Criteria for terminal and canonical singularities.

Let  $\mathbb{C}$  be the field of complex numbers, and  $\mathbb{C}\{x\}$  denotes the ring of convergent power series in variables  $x$ .

Lemma 1. Let  $(X, p)$  be the germ of an  $n$ -dimensional terminal (resp. canonical) singularity of index  $m$ . Let  $(X', p')$  be the germ of an  $n$ -dimensional reduced Gorenstein variety and  $f : (X', p') \rightarrow (X, p)$  a morphism such that  $f$  factors as

$$X' \xrightarrow{g} Y \xrightarrow{h} X,$$

where  $h$  is a blow-up of  $X$  and  $g$  is quasi-finite. Let  $\omega$  be a generator of  $\omega_X^{(m)}$  at  $p$ . Then  $f^*\omega$ , as a meromorphic section of  $\omega_{X'}^{\otimes m}$ , vanishes (resp. is regular) along an arbitrary irreducible divisor  $D \ni p$  such that  $\dim f(D) < n-1$ .

Proof. Let  $D$  be a divisor as in the lemma. Let  $\pi : \tilde{X}' \rightarrow X'$  be the normalization, and  $X'' \subset \tilde{X}'$  the complement of the singular locus of  $\tilde{X}'$ . Since  $\text{codim}_{\tilde{X}'}(\tilde{X}' - X'') \geq 2$  and since  $(\pi^*\omega_X)|_{X''} \supset \Omega_{X''}^n$ , we may replace  $(X', p')$  by  $(X'', p'')$  for some smooth  $p''$  such that  $\pi(p'') \in D$ . In other words, we may assume that  $X'$  is smooth. Hence in the factorization  $X' \rightarrow Y \rightarrow X$ , we may assume that  $Y$  is normal and  $q = g(p)$  is a smooth point of  $Y$  (by moving  $p$  in  $D$  if necessary). Then  $h^*\omega$  vanishes (resp. is regular) along  $g(D)$ . Since  $g : (X', p') \rightarrow (Y, q)$  is a morphism of manifolds,  $f^*\omega = g^*h^*\omega$  vanishes (resp. is regular) along  $D$ . q.e.d.

Let  $(X, p)$  be a 3-dimensional canonical singularity of index  $m$  such that, for the associated  $\mathbb{Z}_m$ -cover  $\pi : (X', p') \rightarrow (X, p)$  [4,5],  $(X', p')$  is a hypersurface singularity. Then there exist  $\mathbb{Z}_m$ -semi-invariants  $x_1, \dots, x_4$  in the analytic local ring  $\mathcal{O}_{X', p'}^h$  such that

$$(1.1) \quad \rho(x_i) = \zeta^{c_i} x_i \quad (i = 1, \dots, 4)$$

$$(1.2) \quad \mathcal{O}_{X', p'}^h \cong \mathbb{C}\{x_1, \dots, x_4\}/(\mathfrak{P}),$$

where  $\rho$  (resp.  $\zeta$ ) is a generator of  $\mathbb{Z}_m$  (resp.  $\mu_m$ ),  $c = (c_1, \dots, c_4) \in \mathbb{Z}^4$  be such that  $\gcd(c, m) = \text{g.c.d.}\{c_1, \dots, c_4, m\} = 1$ , and  $\mathfrak{P}$  is a semi-invariant. Since  $\pi$  is unramified outside  $p'$ , one has

$$\{q \in X' \mid x_i(q) = 0 \text{ if } c_i \not\equiv 0 \pmod{d}\} = \{p'\}$$

for any divisor  $d > 1$  of  $m$ . Hence, for any divisor  $d > 1$  of  $m$ , one has

$$(1.3) \quad \gcd(c, m) = 1, \quad \#\{i \in [1, 4] \mid c_i \equiv 0 \pmod{d}\} \leq 1.$$

Now we reverse the process:

Notation (2.0). Let  $\mathbb{Z}_m$  act on  $\mathbb{C}\{x_1, \dots, x_4\}$  by (1.1) with  $\rho$  (resp.  $\zeta$ ) a generator of  $\mathbb{Z}_m$  (resp.  $\mu_m$ ), where  $c \in \mathbb{Z}^4$  satisfies (1.3) for an arbitrary divisor  $d > 1$  of  $m$ . Let  $\mathfrak{P}$  be a semi-invariant of  $\mathbb{C}\{x_1, \dots, x_4\}$  such that  $\mathbb{C}\{x_1, \dots, x_4\}/(\mathfrak{P})$  is normal, and let  $(X', p')$  be the germ of a hypersurface at 0 defined by (1.2), and  $(X, p) = (X', p')/\mu_m$ .

Then we have

Theorem 2. Under Notation (2.0), let  $\sigma$  be an arbitrary element of  $\mathbf{Z}_m$ , and  $a = (a_1, \dots, a_4)$  a 4-tuple of arbitrary integers  $\geq 0$  such that  $\sigma(x_i) = \zeta^{a_i} x_i$  ( $i = 1, \dots, 4$ ) and that at least three of  $a_1, \dots, a_4$  are positive. Let  $e(a) = \max \{j \mid \varphi(x_1 t^{a_1}, \dots, x_4 t^{a_4}) \equiv 0 \pmod{t^j}\}$  and  $|a| = a_1 + \dots + a_4$ . Then if  $(X, p)$  is terminal (resp. canonical), then  $|a| - m - e(a) > 0$  (resp.  $\geq 0$ ).

Proof. (2.1) Let  $X_0'$  be  $\mathbf{C}^4$  with global coordinates  $x_1, \dots, x_4$ , and let  $\mathbf{Z}_m$  act on  $X_0'$  by (1.1). Then  $T' = (\mathbf{C}^*)^4 \subset \mathbf{C}^4 = X_0'$  is an affine torus embedding, and to the affine torus embedding  $T' \subset X_0'$  correspond the group  $\Gamma(T') \cong \mathbf{Z}^4$  of 1 parameter subgroups of  $T'$  and a cone  $C(X_0')$  of  $\Gamma(T') \otimes \mathbf{Q}$ . Let  $\Gamma(T') = \mathbf{Z}^4$  and  $C(X_0') = \mathbf{Q}_+^4$  in the standard way, where  $\mathbf{Q}_+ = \{q \in \mathbf{Q} \mid q \geq 0\}$ . Then, to  $T = T'/\mathbf{Z}_m \rightarrow X_0 = X_0'/\mathbf{Z}_m$ , correspond  $\Gamma(T) = \mathbf{Z}^4 + \mathbf{Z} c/m$  and  $C(X_0) = \mathbf{Q}_+^4$ . By the definition of  $a$ , there exist integers  $\beta, \tau$  such that  $\beta c_i \equiv \tau a_i \pmod{m}$ , where  $\tau$  is prime to  $m$ . Hence  $a/m \in \Gamma(T)$ .

(2.2) Let  $\varphi'$  be a convergent power series defined by

$$\varphi'(w, y_1, \dots, y_4) = w^{-e(a)} \cdot \varphi(y_1 w^{a_1}, \dots, y_4 w^{a_4}).$$

Then  $\varphi'(0, y)$  is a non-zero weighted homogeneous polynomial of weight  $e(a)$  in  $y_1, \dots, y_4$  for weight  $y_i = a_i$  (cf. (\*) below). Since  $y_1 y_2$  and  $y_3 y_4$  are coprime,  $\varphi'(0, y)$  has a prime factor which is prime to  $y_1 y_2$  or  $y_3 y_4$ . By symmetry, we may assume that  $\varphi'(0, y)$  has a prime factor prime to  $y_1 y_2$  and  $a_1 > 0$ . Since

$$(*) \quad \varphi'(0, y_1 t^{a_1}, \dots, y_4 t^{a_4}) = t^{e(a)} \varphi'(0, y_1, \dots, y_4),$$

one can find  $r_2, r_3, r_4 \in \mathbb{C}$  such that

$$\varphi'(0, 1, r_2, r_3, r_4) = 0.$$

(2.3). Let  $e_2 = (0, 1, 0, 0)$ ,  $e_3 = (0, 0, 1, 0)$ ,  $e_4 = (0, 0, 0, 1) \in \mathbb{Z}^4$ , and let  $C$  be the cone spanned by  $a$ ,  $e_2, e_3, e_4$  in  $\mathbb{Q}^4$  and  $M = \mathbb{Z} e_2 \oplus \mathbb{Z} e_3 \oplus \mathbb{Z} e_4 \subset \mathbb{Q}^4$ . Then the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} a \oplus M & \longrightarrow & \mathbb{Z} \frac{a}{m} \oplus M \\ \downarrow & & \downarrow \\ \mathbb{Z}^4 & \longrightarrow & \mathbb{Z}^4 + \mathbb{Z} \frac{C}{m} \end{array}$$

gives a commutative diagram of tori:

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

and  $C \subset \Gamma(R') \otimes \mathbb{Q}$  gives a torus embedding

$$R' \subset Z_0' = \text{Spec } \mathbb{C}[\bar{w}, \bar{x}_2, \bar{x}_3, \bar{x}_4],$$

where  $a, e_2, e_3, e_4$  of  $\Gamma(R')$  correspond to  $w, \bar{x}_2, \bar{x}_3, \bar{x}_4$ . Then

$$R \subset Z_0 = \text{Spec } \mathbb{C}[\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4],$$

with  $\bar{x}_1 = w^m$ , is the torus embedding corresponding to

$C \subset \Gamma(R) \otimes \mathbb{Q}$ , and commutative diagrams

$$\begin{array}{ccccc} \Gamma(R') \otimes \mathbb{Q} & \longrightarrow & \Gamma(R) \otimes \mathbb{Q} & \longrightarrow & \Gamma(T) \otimes \mathbb{Q} \\ \cup & & \cup & & \cup \\ C & \longrightarrow & C & \longrightarrow & \mathbb{Q}_+^4 \end{array}$$

and

$$\begin{array}{ccccc} \Gamma(R') \otimes \mathbb{Q} & \longrightarrow & \Gamma(T') \otimes \mathbb{Q} & \longrightarrow & \Gamma(T) \otimes \mathbb{Q} \\ \cup & & \cup & & \cup \\ C & \longrightarrow & \mathbb{Q}_+^4 & \longrightarrow & \mathbb{Q}_+^4 \end{array}$$

give a commutative diagram

$$\begin{array}{ccc} Z_0' & \longrightarrow & Z_0 \\ f' \downarrow & & f \downarrow \\ X_0' & \longrightarrow & X_0 \end{array}$$

where  $f'$  is given by

$$x_1 = w^{a_1}, x_2 = \bar{x}_2 w^{a_2}, x_3 = \bar{x}_3 w^{a_3}, x_4 = \bar{x}_4 w^{a_4},$$

If  $T \subset Y_0$  is the torus embedding corresponding to  $C \subset \Gamma(T) \otimes \mathbb{Q}$ , then  $Z_0 \rightarrow Y_0$  is finite and  $Y_0 \rightarrow X_0$  is a blow-up.

(2.4). Let  $s' = V(w, \bar{x}_2^{-r_2}, \bar{x}_3^{-r_3}, \bar{x}_4^{-r_4}) \in Z_0'$  (resp.  $s$  = the image of  $s'$  in  $Z_0$ ), and let

$$\psi(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = w^{-e(a)} \cdot \varphi(w^{a_1}, \bar{x}_2 w^{a_2}, \bar{x}_3 w^{a_3}, \bar{x}_4 w^{a_4}),$$

(note that the right hand side is a holomorphic function in

$\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$  defined near  $s$ .) Let  $(Z, s) \subset (Z_0, s)$  (resp.  $(Z', s') \subset (Z'_0, s')$ ) be defined by  $\psi = 0$ . Then  $f$  (resp.  $f'$ ) induces  $f : (Z, s) \rightarrow (X, p)$  (resp.  $f' : (Z', s') \rightarrow (X', p')$ ), which satisfies the conditions of Lemma 1 by (2.3). We have the following commutative diagram of natural morphisms:

$$\begin{array}{ccc} Z' & \xrightarrow{\tau} & Z \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{\pi} & X. \end{array}$$

By Poincaré residue formula,

$$\omega_{X'} = \text{Res} \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\wp} = - \frac{dx_1 \wedge dx_2 \wedge dx_3}{\wp_4},$$

$$f'^* \omega_{X'} = - \frac{d(w^{a_1}) \wedge d(\bar{x}_2 w^{a_2}) \wedge d(\bar{x}_3 w^{a_3})}{f'^* \wp_4},$$

where  $\wp_4 = \partial \wp / \partial x_4$ . By calculation, one sees

$$\begin{aligned} & d(w^{a_1}) \wedge d(\bar{x}_2 w^{a_2}) \wedge d(\bar{x}_3 w^{a_3}) \\ &= a_1 \cdot w^{a_1+a_2+a_3-1} \cdot dw \wedge d\bar{x}_2 \wedge d\bar{x}_3. \end{aligned}$$

Since  $f'^* \wp = w^{e(a)} \psi$ , it follows from the chain rule that

$$(f'^* \wp_4) \cdot w^{a_4} = w^{e(a)} \psi_4,$$

where  $\psi_4 = \partial \psi / \partial \bar{x}_4$ . Thus

$$f'^* \omega_{X'} = - a_1 \cdot w^{|a|-e(a)-1} \cdot \frac{dw \wedge d\bar{x}_2 \wedge d\bar{x}_3}{\psi_4}.$$



One also has

$$\omega_Z^* = \text{Res} \frac{d\omega \wedge d\bar{x}_2 \wedge d\bar{x}_3 \wedge d\bar{x}_4}{\psi} = - \frac{d\omega \wedge d\bar{x}_2 \wedge d\bar{x}_3}{\psi_4}.$$

One has

$$f^* \omega_X = a_1 \cdot \omega^{|a|-e(a)-1} \omega_Z.$$

By construction, we have

$$\tau^* \omega_Z = m \cdot \omega^{m-1} \omega_Z.$$

Hence we have

$$f^* \omega_X = (a_1/m) \cdot \omega^{|a|-e(a)-m} \tau^* \omega_Z.$$

Since  $\pi : X' \rightarrow X$  is unramified in codimension 1 by (1.3), one has  $\pi^* \omega_X^{(m)} = (\text{unit}) \cdot \omega_X^{\otimes m}$  near  $p'$  (by abuse of language,  $\omega_X^{(m)}$  denotes one of its generators at  $p$ ), and

$$f^* \omega_X^{(m)} = (\text{unit}) \cdot \bar{x}_1^{|a|-e(a)-m} \omega_Z^{\otimes m} \text{ near } s.$$

Since  $\{\bar{x}_1 = 0\}$  is a divisor through  $s$  collapsed by  $f$ , one has  $|a| - e - m > 0$  (resp.  $\geq 0$ ). q.e.d.

Under Notation (2.0), let  $x_1, \dots, x_r \in \mathbb{C}\{x_1, \dots, x_4\}$  ( $r \geq 2$ ) be  $\mathbb{Z}_m$ -semi-invariants with the same character such

that  $x_i \mathbb{C}\{x_1, \dots, x_4\} = u_i \mathbb{C}\{x_1, \dots, x_4\}$  for some monomial  $u_i$  in  $x_1, \dots, x_4$  ( $i = 1, \dots, r$ ) and that  $u_1, \dots, u_r$  are linearly independent over  $\mathbb{C}$  and the locus defined by  $x_1 = \dots = x_r = 0$  is of dimension  $\leq 1$ . Let  $\Phi$  be the linear system generated by  $x_1, \dots, x_r$ , and assume that our  $\mathcal{P}$  is written as

$$\mathcal{P} = \sum_{i=1}^r \lambda_i x_i$$

for some  $\lambda = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r$ . By Bertini's theorem,  $\mathcal{P} = 0$  defines a normal variety for general  $\lambda$  since the base locus of  $\Phi$  is of dimension  $\leq 1$ . Let  $\sigma, a, \zeta$  be as in Theorem 2, then the value of  $e(a)$  given in Theorem 2 does not depend on the choice of  $\lambda \in (\mathbb{C}^*)^r$ . Then under the notation of Theorem 2, the following is the corollary to the proof of Theorem 2.

**Corollary 2.1.** If  $|a| - m - e(a) > 0$  (resp.  $\geq 0$ ) for arbitrary  $\sigma, a, \zeta$  as in Theorem 2, then  $(X, p)$  is terminal (resp. canonical) for general  $\lambda$ . If  $(X, p)$  has an isolated singular point at  $p$  (resp. canonical singularity outside  $p$ ) for general  $\lambda$ , then one can add the extra conditions  $a_1, \dots, a_4 > 0$  on  $a$  in the statement above.

**Proof.** Let  $X_0 = \mathbb{C}^4 / \mathbb{Z}_m$  with respect to the action given above. Then  $\Phi$  induces a linear system  $\Phi_0$  (of Weil divisors) in a neighborhood of 0 in  $X_0$  which is free from fixed components. By the conditions on  $x_i$ 's, there exists a toric resolution

$h : U_0 \rightarrow X_0$  such that the proper transform  $\psi_0$  of  $\phi_0$  to  $U_0$  is free from base points (principalizer of the coherent sheaf associated to  $\phi_0$ ). Then a general member  $X$  of  $\phi_0$  is normal at 0 and its proper transform  $U = h^{-1}[X]$  is smooth in a neighborhood of  $h^{-1}(0)$ . Thus it is enough to show that  $h^* \omega_X^{(m)}$ , as a meromorphic section of  $\omega_U^{\otimes m}$ , vanishes (resp. is regular) along  $D_0 \cap U$ , where  $D_0 \subset U_0$  is an arbitrary exceptional divisor of  $h$ . We now use the notation of the proof of Theorem 2. Let  $L_+$  be the 1-simplex  $\subset \mathbb{Q}_+^4 \subset \Gamma(T) \otimes \mathbb{Q}$  corresponding to  $D_0$ . Let  $a = (a_1, \dots, a_4)$  be a 4-ple of integers  $\geq 0$  such that  $\mathbb{Z}a/m = \mathbb{Q}L_+ \cap \Gamma(T)$ . By (1.3), the singular locus of  $X_0$  is of dimension  $\leq 1$ , and the base locus of  $\phi_0$  is of dimension  $\leq 1$ . Thus one may assume that the image of  $D_0$  to  $X_0$  is of dimension  $\leq 1$ , whence at least 3 of  $a_1, \dots, a_4$  are positive. Then the notation of (2.3) can be used, and the torus embedding  $T \subset V_0 = T \cup D_0$  corresponds to  $L_+ \subset \Gamma(T) \otimes \mathbb{Q}$ .  $L_+ \subset \Gamma(R) \otimes \mathbb{Q}$  corresponds to the open subset  $W_0$  of  $Z_0$  defined by  $\bar{x}_3 \bar{x}_4 \neq 0$ . Since  $\mathbb{Z}a/m = \mathbb{Q}a/m \cap \Gamma(T)$ ,  $\{\bar{x}_1 = 0\}$  is not in the branch locus of  $W_0 \rightarrow V_0$ . Let  $V, W$  be the proper transforms of  $X$  to  $V_0$  and  $W_0$ , respectively. Then, since  $\psi_0$  is free from base points,  $g : W \rightarrow V$  is unramified over general points of arbitrary irreducible components of  $V \cap D_0$ . Thus by

$$g^* h^* \omega_X^{(m)} = (\text{unit}) \cdot \bar{x}_1^{-|a| - e(a) - m} \omega_W^{\otimes m} \text{ along } W \cap$$

$D_0$ ,

(2.4), we have corollary 2.1.

q.e.d.

Corollary 2.2. Under the assumptions of Theorem 2, assume that  $m$  is odd, and

$$\varphi = x_1^2 + f(x_2, x_3, x_4) \quad (f \in \mathbb{C}\{x_2, x_3, x_4\}).$$

Let  $n = \max\{j \mid \varphi(x_2 t^{a_2}, x_3 t^{a_3}, x_4 t^{a_4}) \equiv 0 \pmod{t^j}\}$ . Then

$$a_2 + a_3 + a_4 > (\text{resp. } \geq) \begin{cases} m + n/2 & \text{if } n \text{ is even} \\ m/2 + n/2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. One has  $2 \cdot \text{wt } x_1 \equiv n \pmod{m}$ . If  $n$  is even, then we choose  $a_1 = n/2$ , keeping  $a_2, a_3, a_4$  the same. Then  $n/2 + a_2 + a_3 + a_4 > (\text{resp. } \geq) m + n$ . If  $n$  is odd, then we choose  $a_1 = (m+n)/2$ , keeping  $a_2, a_3, a_4$  the same. Then  $(n + m)/2 + a_2 + a_3 + a_4 > (\text{resp. } \geq) m + n$ .  $\text{q.e.d.}$

For the approximation of  $\varphi$ , we need the following which easily follows by the argument of [1]:

Theorem 3. Let  $\varphi \in \mathbb{C}\{x, y, z, u\}$ . Assume that  $\varphi$  has an isolated singular point at  $(0)$ , and that  $\mathbb{Z}_m$  acts on  $\mathbb{C}\{x, y, z, u\}$  in such a way that  $x, y, z, u$ , and  $\varphi$  are semi-invariants. Then for an arbitrary integer  $b > 0$ , there exists an integer  $n > 0$  such that, for an arbitrary semi-invariant  $\psi \in \mathbb{C}\{x, y, z, u\}$  with the property  $\psi \equiv \varphi(x, y, z, u)^n$ , there exists an analytic  $\mathbb{C}$ -automorphism  $\sigma$  of  $\mathbb{C}\{x, y, z, u\}$  commuting with  $\mathbb{Z}_m$ -action (will be called a  $\mathbb{Z}_m$ -automorphism, for short) such that  $\sigma(\varphi)\mathbb{C}\{x, y, z, u\} = \psi\mathbb{C}\{x, y, z, u\}$ ,  $\sigma \equiv \text{id. modulo } (x, y, z, u)^b$ .

Corollary 4. Under the notation and the assumptions of Theorem 3, if  $\mathfrak{P} \in (x, y, z, u)^2$  and if  $xy$  appears in  $\mathfrak{P}$ , then there exist a  $\mathbb{Z}_m$ -automorphism  $\sigma$  of  $\mathbb{C}\{x, y, z, u\}$  such that

$$\sigma(x) - x \in (y, z, u) + (x, y, z, u)^2,$$

$$\sigma(y) - y \in (x, z, u) + (x, y, z, u)^2,$$

$$\sigma(z) - z, \sigma(u) - u \in (x, y, z, u)^2,$$

$$(\sigma(\mathfrak{P})) = (xy + f(z, u)) \text{ as ideals for some } f \in \mathbb{C}\{z, u\}.$$

## §2. Notation and classification.

Let  $\mathcal{P}$  be an element of  $(x, y, z, u)^2 \mathbb{C}\{x, y, z, u\}$  which has a  $\mathbb{Z}_m$ -action ( $m > 1$ ) such that  $x, y, z, u, \mathcal{P}$  are semi-invariants. Assume that  $\mathcal{P}$  has an isolated cDV singularity at the origin  $(0)$ , that the quotient of  $\{\mathcal{P} = 0\}$  by  $\mathbb{Z}_m$  has a terminal singularity at  $(0)$ , and that the action of  $\mathbb{Z}_m$  is free on  $U - (0)$ , where  $U \ni (0)$  is an open set of  $\{\mathcal{P} = 0\}$ . By a  $\mathbb{Z}_m$ -automorphism, we mean an analytic  $\mathbb{C}$ -automorphism of  $\mathbb{C}\{x, y, z, u\}$  commuting with  $\mathbb{Z}_m$ -action unless otherwise mentioned. We will keep these assumptions and notation, unless otherwise mentioned.

Fixing a primitive  $m$ -th root  $\zeta$  of 1, and given the  $\mathbb{Z}_m$ -action above, we associate to each  $\sigma \in \mathbb{Z}_m$  a weight modulo  $m$  (denoted by  $\sigma\text{-wt} \pmod{m}$ );  $\sigma\text{-wt}(x) \equiv a(\sigma)$ , ...,  $\sigma\text{-wt}(u) \equiv d(\sigma) \pmod{m}$  are determined by  $\sigma(x) = \zeta^{a(\sigma)} \cdot x$ , ...,  $\sigma(u) = \zeta^{d(\sigma)} \cdot u$ . If  $\sigma$  is a generator of  $\mathbb{Z}_m$ , we may simply call  $\sigma\text{-wt}$  a wt., if there is no danger of confusion. By Theorem 2 and Corollary 2.2, one can almost classify such  $\mathcal{P}$  as above. The results are as follows.

Theorem 5. If  $xy$  appears in  $\mathcal{P}$ , then one of the following holds (after exchanging  $z, u$  if necessary):

(1)  $\text{wt } x + \text{wt } y \equiv \text{wt } z \equiv 0 \pmod{m}$ , and  $\text{wt } u, \text{wt } x, \text{wt } y$  are prime to  $m$ .

(2)  $m = 4$  and there exists a generator  $\sigma$  of  $\mathbb{Z}_4$  such that  $\sigma\text{-wts}$  of  $x, y, z, u$  are  $1, 1, 2, 3 \pmod{4}$ .

By Theorem 5, one can easily prove

Lemma 6. If  $x^2$  and  $y^2$  appear in  $\mathcal{P}$  and  $m = 2$ , then

(1) if  $\text{wt } x \equiv \text{wt } y \pmod{2}$ , then after exchanging  $z, u$  (if necessary), one has wts of  $x, y, z, u$  equal to  $1, 1, 0, 1 \pmod{2}$ , and modulo  $\mathbf{Z}_2$ -automorphism,  $(\mathcal{P}) = (x^2 + y^2 + f(z, u^2))$  for some  $f \in \mathbf{C}\{z, u\}$ ,

(2) if  $\text{wt } x \not\equiv \text{wt } y \pmod{2}$ , then after exchanging  $x, y$  (if necessary), one has wts of  $x, y, z, u$  equal to  $1, 0, 1, 1 \pmod{2}$ , and modulo  $\mathbf{Z}_2$ -automorphism,  $(\mathcal{P}) = (x^2 + y^2 + f(z, u))$  for some  $f \in \mathbf{C}\{z, u\}$ .

Thus one gets

Theorem 7. If the quadratic part  $\mathcal{P}_2$  of  $\mathcal{P}$  has rank  $\geq 2$  as a quadratic form, then after permutation of  $x, y, z, u$  (if necessary), one of the following holds.

(1)  $\text{wt } x + \text{wt } y \equiv 0$ ;  $\text{wt } z \equiv \text{wt } \mathcal{P} \equiv 0$ ;  $\text{wt } x, \text{wt } y, \text{wt } u$  are prime to  $m$ , and modulo  $\mathbf{Z}_m$ -automorphism,  $(\mathcal{P}) = (xy + f(z, u^m))$  for some  $f \in \mathbf{C}\{z, u^m\}$ .

(2)  $m = 4$ : wts of  $x, y, z, u, \mathcal{P}$  are  $1, 3, 2, 1, 2 \pmod{4}$  (after choosing a generator of  $\mathbf{Z}_4$ ), and modulo  $\mathbf{Z}_4$ -automorphism,  $(\mathcal{P}) = (x^2 + y^2 + f(z, u^2))$  for some  $f \in \mathbf{C}\{z, u^2\}$ .

(3)  $m = 2$ : wts of  $x, y, z, u, \mathcal{P}$  are  $1, 0, 1, 1, 0 \pmod{2}$ , and modulo  $\mathbf{Z}_2$ -automorphism,  $(\mathcal{P}) = (x^2 + y^2 + f(z, u))$  for some  $f \in (z, u)^4 \mathbf{C}\{z, u\}$ .

Remark 7.1. In case (1) (resp. (2), (3)) of Theorem

7, if  $f$  is a general linear combination of a finite number of monomials in  $(z^2, u^m)\mathbb{C}\{z, u^m\}$  (resp.  $(z^3, u^2)\mathbb{C}\{z^2, zu^2, u^4\}$ ,  $(z^4, z^3u, z^2u^2, zu^3, u^4)\mathbb{C}\{z^2, zu, u^2\}$ ) and if  $f$  has an isolated singular point at 0, then  $(X, p)$  is a terminal singularity. This follows from Corollary 2.1.

Lemma 8. If  $\text{rk } \mathfrak{P}_2 \leq 1$  and  $u^2$  appears in  $\mathfrak{P}_2$ , then one has

$(\mathfrak{P}) = (u^2 + f(x, y, z))$  modulo  $\mathbb{Z}_m$ -automorphism, where  $f \in (x, y, z)^3\mathbb{C}\{x, y, z\}$  has non-zero cubic term  $f_3$ .

This follows from the definition of cDV points.

Theorem 9. Under the situation of Lemma 8, if  $f_3$  is not a cube of a linear factor, then after permutation of  $x, y, z$  and choice of generator  $\sigma$  of  $\mathbb{Z}_m$ , one of the following holds.

(1)  $m = 2$ , wts of  $x, y, z, u, \mathfrak{P}$  are  $1, 1, 0, 1, 0 \pmod{2}$ , and  $xyz$  or  $y^2z$  appears in  $f_3$ .

(2)  $m = 3$ ,  $\sigma$ -wts of  $x, y, z, u, \mathfrak{P}$  are  $1, 2, 2, 0, 0 \pmod{3}$ , and  $f_3 = x^3 + y^3 + z^3$ ,  $x^3 + yz^2$ , or  $x^3 + y^3$  modulo  $\mathbb{Z}_3$ -automorphism of  $\mathbb{C}\{x, y, z\}$ .

Remark 9.1. In cases (1), (2) of Theorem 9, it is easy to see that modulo  $\mathbb{Z}_m$ -automorphism and units of  $\mathbb{C}\{x, y, z, u\}$ , one may put  $\mathfrak{P}$  in one of the following forms by Theorem 3:

Case 1.  $m = 2$ , wts of  $x, y, z, u$  are  $1, 1, 0, 1 \pmod{2}$ .



2,

$$(9.1.1) \quad \varphi = u^2 + xyz + x^{2a} + y^{2b} + z^c,$$

$$(9.1.2) \quad \varphi = u^2 + y^2z + \lambda yx^{2a-1} + g,$$

$$(a, b \geq 2, c \geq 3, \lambda \in k, g \in (x^4, x^2z^2, z^3)\mathbb{C}\{x^2, z\}),$$

Case 2.  $m = 3$ , wts of  $x, y, z, u$  are 1, 2, 2, 0 mod.

3,

$$(9.2.1) \quad \varphi = u^2 + x^3 + y^3 + z^3,$$

$$(9.2.2) \quad \varphi = u^2 + x^3 + yz^2 + \lambda y^{3r+1}x + \mu y^{3s},$$

$$(9.2.3) \quad \varphi = u^2 + x^3 + y^3 + xyz^3 \cdot \alpha + xz^4 \cdot \beta + yz^5 \cdot \gamma + z^6 \cdot \delta,$$

$$(r \geq 1, s \geq 2, \lambda, \mu \in k, \alpha, \beta, \gamma, \delta \in \mathbb{C}\{z^3\}).$$

For (9.1.1) and (9.2.1),  $(X, p)$  is terminal. If  $\varphi$  is a general linear combination of a finite number of monomials as in (9.1.2), (9.2.2), or (9.2.3), and if  $\varphi$  has an isolated singularity at 0, then  $(X, p)$  is terminal.

Lemma 10. Under the situation of Lemma 8, if  $f_3 = x^3$ , then modulo  $\mathbb{Z}_m$ -automorphism,

$$(\varphi) = (u^2 + x^3 + g(y, z)x + h(y, z)),$$

where  $g, h \in \mathbb{C}\{y, z\}$ ,  $g \in (y, z)^3$ ,  $h \in (y, z)^4$ .

Theorem 11. Under the situation of Lemma 10, one has  $m = 2$  and wts of  $x, y, z, u, \varphi$  are 0, 1, 1, 1, 0 mod. 2, and  $h \notin (y, z)^5$ .

Remark 11.1. If  $m = 2$  and wts of  $x, y, z, u$  are 0, 1, 1, 1 mod. 2 and if an even polynomial  $g$  (resp.  $h$ ) in  $y, z$  is a general linear combination of a finite number of monomials  $\in (y, z)^4$  and  $h \notin (y, z)^5$ , and if

$$\mathcal{P} = u^2 + x^3 + g(y, z)x + h(y, z)$$

has an isolated singularity at 0, then  $(X, \mathcal{P})$  is terminal.

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